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ON THE CURVATURE PRESERVING PIECEWISE APPROXIMATION OF CLOSED PLANAR CURVES BY MINMAXION

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ABSTRACT

The problem of fitting curves to observational data is quite old and continues to evince interest. The task is challenging when the genesis of the data is unknown. This leads to the complex problem of ordering the points and then fitting curves to these ordered points. The problem is further compounded when the points lie along non function-like curves, closed curves being a special case. This problem has been addressed earlier by us with respect to open, non function-like curves. In the present study, we solve the problem by first sampling the points along the target shape (a closed curve), ordering the sampled points, segmenting the ordered points at significant points and subsequently approximate each segment parametrically. The approach is data guided in which the entire process, right from sampling, ordering, selecting the significant points and fitting curves to each segment is fully automated. Data determines the degree of each curve segment in such a way that the first and second derivatives at junction points match giving cubic spline smoothness to the whole fitting process. The technique has been applied on some test curves and results appear encouraging. Results on one test curve are presented.

KEYWORDS: Curve Parametrization, Curve Segmentation, Knot Selection, Minaddition, Minmaxion, Ordering of Points, Ordering Index

1. INTRODUCTION

Object recognition is one of the key issues in image analysis like general computer vision, military target recognition and biometrics. In many applications, image analysis can be reduced to the analysis of shapes. To describe shape through object boundary is a preliminary but critical step. Describing the boundary of an object reduces to the problem of curve or surface fitting.

Curves have a striking visual nature. Curves that need to be fit arise in several contexts like in cartography and biology, to name a few. Giving analytical representation to these poses several challenges. This includes taking cognizance of noise and selection of a suitable fitting strategy. The problem can be stated as "*given a set of 'n' data points* (x_i, y_i), *i = 1: n taken from a target curve, reconstruct a curve which approximates the original curve to a satisfactory extent and which also pleases the eye*". Observational data are usually subject to measurement errors and hence approximation techniques may be preferred to interpolation techniques. This enables us to avoid unwanted undulations. We are concerned with analytical representation of simple closed curves by segmenting the target curve at meaningful points.

In the literature, many papers have dealt with the problem of curve fitting assuming that data are ordered. But ordering points by itself is a non-trivial problem particularly when the genesis of the data is unknown. The issue of ordering the points has been addressed in [13]. Matrix operations called *minmaxion* and *minaddition* [14, 15, 16, 17, 18] have been adopted successfully to achieve two goals (i) ordering of the points and (ii) defining a parametrization scheme.

Non function like curves can be represented effectively using curve parametrization. A survey of existing methods for curve parametrization is briefly presented. When data points are already ordered, uniform parametrization, the simplest of all, has been tried but with less success. This is because this technique creates singularities like corners. A better strategy is to adopt chord length parametrization because chord length is actually an approximation of the true length of the fitted curve. Circular arc parametrization is yet another alternative in which the arc length is estimated by fitting a circle through each group of three consecutive points. [1, 2, 3, 4, 5, 6] can be referred for a survey of available parametrization techniques. One can also refer to [7, 8, 9, 10, 11]. We have a slightly different approach to the problem of curve parametrization.

The paper is organized as follows

In section II, the concepts of *minmaxion* and *minaddition* along with satiety have been explained leading to ordering of points and inducing curve parametrization. This is preceded by a strategic sampling technique to select data points along the target curve. In section III, curve segmentation strategies have been discussed. Segmentation of the curve at points where radii of curvature are a minimum along the curve has been chosen as the basis for knot selection. In section IV, the detailed fitting strategy with essential statistical analysis has been presented. Section V deals with conclusions. Section VI deals with future scope of study and Section VII cites references used in this study.

(2) 2.1. SAMPLING OF DATA POINTS

The target curve, a simple closed curve in this study, when digitized issues a dense set of points. We need to sample the points so as to retain most of the features but at the same time keep the number of selected points not too large. Otherwise the matrix computations that involve finding inter-node distances, minmaxion and minaddition matrices become unmanageable. We first fix a threshold distance ‘2’. Consider the first point as the source point. We find distances from the source point to all other points including the source point itself. Select all distances which are less than the threshold ‘2’. All such points can be treated as a cluster. Take points in this cluster away from the original set of points and find similar clusters as explained above till the list is exhausted. Take the geometric centers of each cluster. These centers form the sampling points. If the number of these cluster centers is too large, one can raise the threshold level iteratively till we accumulate cluster centers which are not too big in number.

2.2. Minmaxion and Minaddition

Pandit [14, 15, 16] visualized these operations particularly in the context of cluster analysis. Pandit and Ramamurthy [13] applied these operations in devising a new curve parametrization scheme.

Definition: MINMAXION

$$C \text{ is the min-max product of } A \text{ and } B \quad C \triangleq A \otimes B \quad \text{where } c_{ij} = \min_{x} \{\max(a_{ix}, b_{xj})\}$$

Definition: MINADDITION

C is the min-ad product of A and B $\qquad C \triangleq A \oplus B \quad where\ c_{ij} = \min_x \{max\,(a_{ix} + b_{xj})\}$

Satiated Matrices

If A is a zero diagonal matrix of order n, and $A^{k+1} = A^k$ for some positive integer, $k<n$, then A^k is the satiated matrix of A. In fact, one can define *satiated minmaxion* and *satiated minaddition* even when the zero-diagonal matrix $D = [d_{ij}]$ is not symmetric. The satiated minmaxion and minaddition are exploited in inducing an ordering among points and defining the parameter for curve fitting.

2.3 Ordering the Points and Parametrization

Consider a test curve on which we take a discrete set of points. Once we have the co-ordinates of the point-pairs, we can compute inter-node distances d_{ij} (say, Euclidean distances) and store these distances in the distance matrix D.

$$D = [d_{ij}] = \begin{cases} 0\ , & i = j \\ > 0\ , & i \neq j \end{cases}$$

Let $S = D^*$ be the minmax satiated matrix of D, i.e. $D^* = D^r = D^{r+1}$ for some $r < n$. The element d^*_{ij} of D^*gives the (r^{th} order) connective distance from i to j. Each of these paths will have a link of largest length. Then d^*_{ij} is the smallest among these largest links in the different paths. Let p^*_{ij} be the number of steps from i to j along this optimal path. The number and the actual path itself can be obtained by the use of minaddition. One can now define the Direct Link Matrix P from the matrix S as follows.

$$P = [p_{ij}] = \begin{cases} 0 & i = j, \\ 1 & d_{ij} = s_{ij}, \quad i \neq j \\ \infty & otherwise \end{cases}$$

The *minad* satiated matrix of P, denoted by P^* called the *step length matrix*, gives the number of steps between point-pairs along these paths. Choosing a point-pair with largest step length, say α to β, one gets the path from α to β on which a relatively large number of points lie in an ordered fashion; the number of steps between any point-pair along this path will be less than this number and one can take this path as an arterial path along which many points lie in a well-defined sequence. If it so happens that, $P^*_{\alpha\beta} = (n-1)\ or\ P^*_{\alpha\beta} \sim (n-1)$ one may infer that nodes $\alpha\ to\ \beta$ are the end points of a long connective path. Since the sequence of points between $\alpha\ to\ \beta$ is now available, one can accept this sequence of points along this path as the appropriate ordering among the n points. Ordering of points along a curve, in general a difficult problem by itself has now been addressed, particularly in the case of open curves. In the case of simple closed curves, one can segment the curve in to segments by a proper selection of knots. What remains to be tackled is curve parametrization.

The ordering index of the connective path itself was proposed as the parameter t [13]. The coordinates $x(t_1)\ and\ y(t_2)$ can now be fitted as functions of 't'.

3. CURVE SEGMENTATION

A global fit to the data using a single curve is not productive particularly when there are important features along the ordered data points. To fit curves to a set of irregularly spaced points, one has to (a) partition the data set into subsets (b) a curve should be fitted to the points in each subset. [17] discusses these issues in a particular way. One can refer [13] for a detailed description of the above aspects.

K not locations for curve segmentation in our study are points of maximum curvature (minimum radius of curvature). The formulas to compute the ordinates of the center of the circle of curvature passing through 3 points $A\ (x_1, y_1),\ B\ (x_2, y_2)$ and $C\ (x_3, y_3)$ joined by lines with slopes m_1 *and* m_2 are given by

$$X_C = \frac{m_1\, m_2\, (y_1 - y_3) + m_2\, (x_1 + x_2) - m_1\, (x_2 + x_3)}{2(m_2 - m_1)} \quad Y_c = -\frac{1}{m_1}\left(X_c - \frac{x_1 + x_2}{2} \right) + \frac{y_1 + y_2}{2}$$

Then the radius of curvature at *B* is its distance from the center of curvature. The choice of the number of curve segments to be used to construct the curve from the given set of points is a non-trivial problem [18]. While too few segments will fail to represent the characteristics of the target curve, using too many segments will make the curve follow noise and introduce many unwanted undulations. Specifically, raising the number of segments will reduce the error between data points and the target curve, but may not necessarily mean that the solution is a better one. A pragmatic choice would be to use as few segments as possible to represent the resulting curve as long as the error between the data points and the curve is within some pre-specified tolerance. In this study, the number of curve segments that are generated is data guided. This is followed by approximating each segment. We have chosen to fit each segment by least squares polynomials ranging from linear to nonic (ninth degree) polynomials achieving continuity up to $C^{(2)}$ at knots. The statistical error sum of squares is the basis for limiting the degree of the approximating polynomial for each segment. That is to say different segments are approximated by the best least squares polynomial which is determined by a pre fixed SSER tolerance (0.5 in our study). Numerical estimates of first and second derivatives are computed using the Newton's divided differences interpolation formula.

4. TEST RESULTS

We consider a simple closed curve as the target curve, to illustrate the technique of ordering the points, locating knot points followed by piecewise polynomial approximations (linear to nonic) along with determination of the error sum of squares (SSER). Table 1 contains sampled data points; Table 2 contains knot point locations based on radii of curvatures Table 3 contains radii of curvatures at these pints.. Table 4 contains information about curve segments based on radii of curvatures. Knot positions are clearly coded in color. Table 5 contains the information about SSER. It can be observed that the SSER reduces significantly as one moves from linear to nonic approximations. Table 6 (a) and (b) give parametric coefficients for linear to nonic polynomial approximations for $X\ (t_1)$ and $Y\ (t_2)$ based on radii of curvature. Table 7 has information about $C^{(1)}$ *and* $C^{(2)}$ continuity at knots. We have consciously kept all matrix computation displays hidden due to the shear largeness of dimension. For a complete study of the technique, one may refer to [13].

Figure 1 is the target curve. Figure 2 shows sampled points with knots determined through radii of curvature. Figure 3 shows curve segments. Figure 4 shows progress of parametric linear to nonic polynomial approximations for the target curve based on the radii of curvature. Figure 5 shows the graphical displays of curve approximations based on radii of curvature.

Figure 1: Target Curve (A Bird in Flight)

Table 1: Sampled Data Points

Points	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
X	191	176	167	158	148	141	136	131	126	122	117	111	105	98	87	75	64	54	41	40	52	62	72	82
Y	86	94	102	111	121	132	143	155	167	179	191	203	213	223	232	239	243	243	244	259	262	267	276	284
Points	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
X	92	105	118	127	133	144	153	159	165	175	188	197	207	217	226	237	246	255	266	276	287	298	309	319
Y	290	294	293	293	289	280	270	259	249	235	237	243	252	260	267	275	282	288	294	301	307	314	316	307
Points	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72
X	335	346	356	368	364	356	348	340	334	328	321	313	305	298	289	281	272	263	257	263	272	282	292	300
Y	306	310	313	311	299	289	280	270	261	252	242	232	222	212	203	195	188	179	168	157	148	141	131	122
Points	73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	
X	309	319	329	340	351	362	377	387	394	383	366	349	333	318	302	287	270	260	253	237	224	213	191	
Y	111	101	91	81	73	65	54	46	36	27	34	41	47	54	61	67	74	75	75	76	79	80	86	

Table 2: K Not Point Locations of Based on Radius of Curvature

Knot locations	1	2	3	4	5	6
Break points	19	33	46	50	66	80

Table 3: Radius of Curvatures at Data Points

Points	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
Radius of curvature	61.5	211	4074	62.27	89.7	394	4074	176	175.6	191.9	163.3	169.8	48.3	89.22	71.47	4074	4074
Points	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34
Radius of curvature	10.74	10.71	53.95	45.90	226.1	91.15	52.3	35.7	4074	4073	109.6	90.94	55.77	294.5	180.8	14.53	27.75
Points	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51
Radius of curvature	83.87	226.1	883.6	387.7	387.7	152.1	131	111	111.1	189.7	31.50	13.98	22.41	33.98	193.2	24.97	8.162
Points	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68
Radius of curvature	36.24	239.4	239.4	136.2	4073	506.6	195	4073	195.4	71.46	4073	91.39	97.08	44.30	13.08	44.30	71.46
Points	69	70	71	72	73	74	75	76	77	78	79	80	81	82	83	84	85
Radius of curvature	75.52	222.8	320.8	142.3	4073	304.8	130	4073	4073	373.9	43.98	9.735	15.94	4073	557.1	216.0	702.3
Points	86	87	88	89	90	91	92	93									
Radius of curvature	526.9	1709	49.08	4073	4073	89.44	89.6	96.5									

Table 4: Curve Segments Based on Radius of Curvature

Segment No.	Curve Segments																		
1	191	176	167	158	148	141	136	131	126	122	117	111	105	98	87	75	64	54	41
	86	94	102	111	121	132	143	155	167	179	191	203	213	223	232	239	243	243	244
2	41	40	52	62	72	82	92	105	118	127	133	144	153	159	165				
	244	259	262	267	276	284	290	294	293	293	289	280	270	259	249				
3	165	175	188	197	207	217	226	237	246	255	266	276	287	298					
	249	235	237	243	252	260	267	275	282	288	294	301	307	314					
4	298	309	319	335	346														
	314	316	307	306	310														
5	346	356	368	364	356	348	340	334	328	321	313	305	298	289	281	272	263		
	310	313	311	299	289	280	270	261	252	242	232	222	212	203	195	188	179		
6	263	257	263	272	282	292	300	309	319	329	340	351	362	377	387				
	179	168	157	148	141	131	122	111	101	91	81	73	65	54	46				
7	387	394	383	366	349	333	318	302	287	270	260	253	237	224	213	191			
	46	36	27	34	41	47	54	61	67	74	75	75	76	79	80	86			

Table 5: SSER for Linear to Nonic Polynomials

Segmt No	SSER for Linear to Nonic Polynomials (Radius olf Curvature)								
1	1222.185	1635.376	160.12	205.9253	37.49128	38.56871	*15.15037*	3.34860498	2.68
2	530.6171	47.56439	21.17074	21.07041	20.88682	20.78199	*20.76844*	1.91E+01	13.6
3	81.20338	60.34586	22.06759	4.524139	0.814219	0.170181	*0*	0	0
4	4.606211	3.232641	1.295577	5.23E-20	0	0	0	0	0
5	194.723	151.3564	138.163	121.8281	120.1376	121.0927	110.8727	104.262668	104
6	517.4323	117.2036	117.6763	110.5642	109.3009	114.3818	111.406	102.986668	94.3
7	2586.324	2526.567	1920.621	1392.711	1400.842	1416.598	1419.92	1427.77308	1331

Table 6: Coefficients of Curve Segment Parametrization for x and y

Coefficients of Linear to 9th Degree Polynomial Curve Approximations for $x(t_1)$								
Segment 1								
Linear	**Quadratic**	**Cubic**	**Quartic**	**Quintic**	**Sextic**	**Septic**	**Octic**	**Nonic**
16.6265 0.8766	2.8405 1.1579 -0.0012	-76.0552 3.7211 -0.0256 0.0001	-35.8872 1.9382 0.0013 -0.0001	49.4260 -2.8479 0.1004 -0.0011	94.9136 -5.9252 0.1817 -0.0021	-226.1343 19.4513 -0.6325 0.0117 -0.0001	-344.2449 30.1125 -1.0345 0.0199 -0.0002	-4.3378 0.4347 -0.0185 0.0004
Coefficients of Linear to 9th Degree Polynomial Curve Approximations for $y(t_2)$								
Segment 1								
Linear	**Quadratic**	**Cubic**	**Quartic**	**Quintic**	**Sextic**	**Septic**	**Octic**	**Nonic**
- 10.4810 1.1229	-65.9688 1.8578 -0.0022	168.2689 -2.9110 0.0282 -0.0001	46.4465 0.4267 -0.0046 0.0001	-15.1210 2.5442 -0.0327 0.0003	249.2060 -8.3860 0.1504 -0.0013	-2.6166 0.1299 -0.0026	5.8824 -0.3387 0.0084 -0.0001	-2.8599 0.1795 -0.0049 0.0001

The parametric equations of linear to 9th degree polynomial approximations for *x* (t_1) and *y* (t_2) for the first segment based on radius of curvature are given below in their corresponding parametric ranges. In the ranges $41 \le t_1 \le 191$, $86 \le t_2 \le 244$,

$$x(t_1) = \begin{cases} 16.6265t_1 + 0.8766 \\ 2.8405t_1^2 + 1.1579t_1 - 0.0012 \\ -76.0552t_1^3 + 3.7211t_1^2 - 0.0256t_1 + 0.0001 \\ -35.8872t_1^4 + 1.9382t_1^3 + 0.0013t_1^2 - 0.0001t_1 \\ 49.4260t_1^5 - 2.8479t_1^4 + 0.1004t_1^3 - 0.0011t_1^2 \\ 94.9136t_1^6 - 5.9252t_1^5 + 0.1817t_1^4 - 0.0021t_1^3 \\ -226.1343t_1^7 + 19.4513t_1^6 - 0.6325t_1^5 + 0.0117t_1^4 - 0.0001t_1^3 \\ -334.2449t_1^8 + 30.1125t_1^7 - 1.0345t_1^6 + 0.0199t_1^5 - 0.0002t_1^4 \\ -4337.8t_1^9 + 0434.7t_1^8 - 18.5t_1^7 + 0.4t_1^6 \end{cases}$$

$$y(t_2) = \begin{cases} -10.4810t_2 + 1.1229 \\ -65.9688t_2^2 + 1.8578t_2 - 0.0022 \\ 168.2689t_2^3 - 2.9110t_2^2 + 0.0282t_2 - 0.0001 \\ 46.4465t_2^4 + 0.4267t_2^3 - 0.0046t_2^2 + 0.0001t_2 \\ -15.1210t_2^5 + 2.5442t_2^4 - 0.0327t_2^3 + 0.0003t_2^2 \\ 249.2060t_2^6 - 8.3860t_2^5 + 0.1504t_2^4 - 0.0013t_2^3 \\ -2616.6t_2^7 + 129.9t_2^6 - 2.6t_2^5 \\ 5882.4t_2^8 - 338.7t_2^7 + 8.4t_2^6 - 0.1t_2^5 \\ -28599t_2^9 + 1795t_2^8 - 49t_2^7 + t_2^6 \end{cases}$$

Table 7: Table Showing $C^{(1)}$ and $C^{(2)}$ Continuity at Knots

$C^{(1)}$ Continuity																		
1	-0.53	-0.89	-1	-1	-1.57	-2.2	-2.4	-2.4	-3	-2.4	-2	-1.67	-1.43	-0.8	-0.6	-0.36	0	-0.1	-15.2
2	-15.2	0.25	0.5	0.9	0.8	0.599	0.307	-0.08	0	-0.67	-0.82	-1.11	-1.83	-1.7	-1.4				
3	-1.4	0.154	0.666	0.9	0.8	0.777	0.727	0.78	0.67	0.54	0.7	0.54	0.64	0.182					
4	0.182	-0.9	-0.06	0.36	0.3														
5	0.3	-0.17	3.008	1.25	1.13	1.252	1.503	1.5	1.43	1.25	1.25	1.43	1	1	0.78	1.001	1.84		
6	1.84	-1.83	-1	-0.7	-1	-1.12	-1.22	-1	-1	-0.91	-0.73	-0.73	-0.73	-0.8	-1.43				
7	-1.43	0.819	-0.41	-0.4	-0.38	-0.47	-0.44	-0.4	-0.4	-0.1	0	-0.06	-0.23	-0.1	-0.3	-0.53			
$C^{(2)}$ Continuity																		
1	0.022	0.049	0.053	0.059	0.131	0.221	0.241	0.267	0.33	0.219	0.167	0.129	0.08	0.036	0.025	0.017	0	0.006	-1.38
2	-1.38	0.011	0.025	0.045	0.04	0.026	0.012	-0	0	-0.04	-0.041	-0.07	-0.15	-0.1	-0.06				
3	-0.06	0.007	0.035	0.045	0.042	0.039	0.036	0.043	0.03	0.026	0.033	0.025	0.029	0.009					
4	0.009	-0.03	-0	0.017	0.014														
5	0.014	-0.02	-0.25	-0.08	-0.07	-0.089	-0.13	-0.12	-0.1	-0.08	-0.083	-0.09	-0.06	-0.06	-0.04	-0.067	183.6		
6	183.6	-0.12	-0.05	-0.03	-0.055	-0.066	-0.06	-0.05	-0	-0.04	-0.033	-0.03	-0.03	-0.05	0.358				
7	0.358	-0.03	0.012	0.012	0.012	0.015	0.014	0.013	0.02	0.006	0	0.002	0.01	0.003	0.007	0.022			

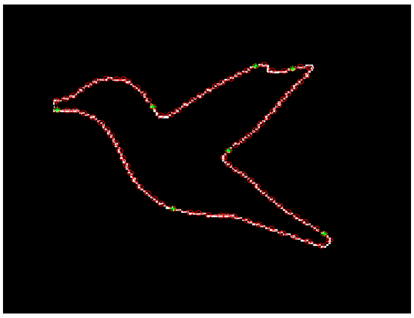

Figure 2: Sampled Data Points Showing Knots

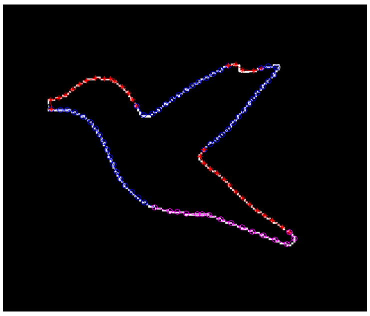

Figure 3: Curve Segments

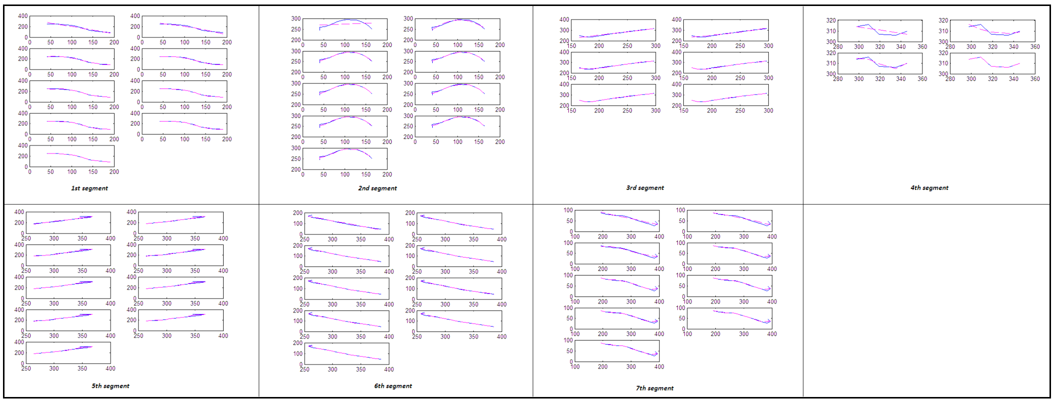

Figure 4: Curve Approximations of Each Segment from Linear to Nonic Best Fitted Polynomial

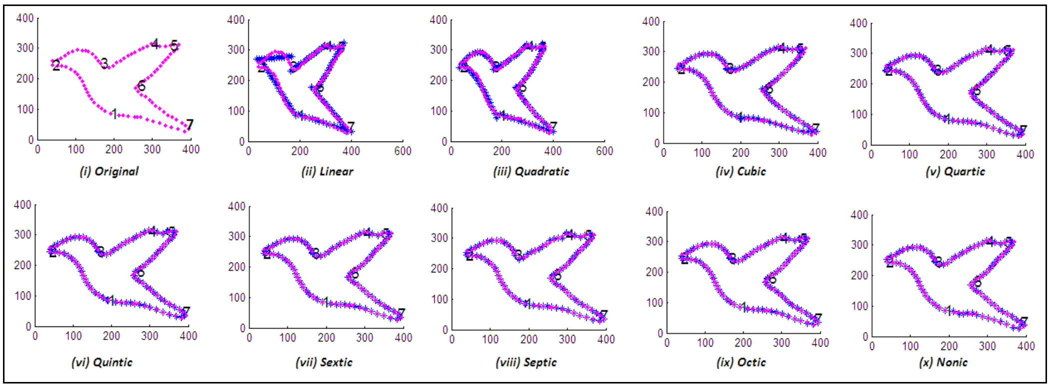

Figure 5: Curve Approximations from Linear to Nonic Polynomials

5. CONCLUSIONS

Data that is collected from field experiments is usually not ordered and sometimes the genesis of the data in unknown. Sampling of the target curve is achieved by a data guided approach. Curve segmentation requires locating knot points and this has been achieved by selecting points where the radius of curvature is a minimum. Location of knots appear in natural positions. Ordering of the points is a critical operation preceding curve approximation. This has been achieved by finding the step-length matrix which gives the number of steps between point pairs along the paths. One can read the ordering sequence from this matrix between a point pair with largest step-length. The ordering index itself is chosen as the parameter 't'. The co-ordinates *x(t)* and *y(t)* are now fitted as function of 't'. The parameter 't' is in the ordinal scale. Each curve segment is approximated from linear to nonic polynomials iteratively. There is a tab on the degree of the approximating polynomial determined by the error sum of squares. $C^{(2)}$ continuity is achieved giving a cubic spline effect to the fitting process. The study reveals that smooth closed curves can be well approximated by this study. Curves with sharp corners have produced mixed results.

6. SCOPE FOR FURTHER STUDY

Other knot selection strategies can be tried. The technique involving *minmaxion* and *minaddition* has been applied by the authors to approximate non function like lineal curves and now extended to simple closed curves by a piecewise fitting strategy. Curves with sharp corners may be the focus of the next study. Though the present study considers data with no noise, future studies can use denoising techniques in the preprocessing stage. Also under study is to use PCA to transform the test curve having a complex shape to PC reference frame where the shape complexity can be reduced.

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